

# Nonuniform Filter Banks: Challenges and Solutions

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# Outline

- ❖ Introduction
- ❖ Perfect Reconstruction of Incompatible Nonuniform Filter Banks
- ❖ Conclusions
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- ❖ Q&A Session



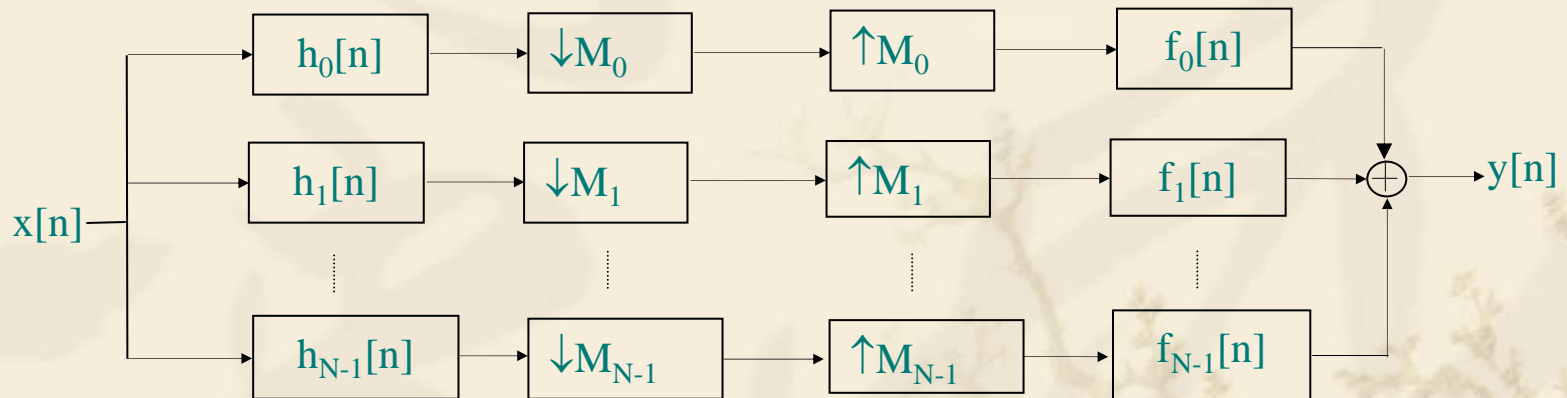
# Introduction

- ❖ Definition of filter banks
- ❖ Advantages of employing filter banks
- ❖ Type of filter banks
- ❖ Incompatible nonuniform filter banks
- ❖ Challenges in filter banks

# Introduction

## ❖ Definition of filter banks

∞ Filter banks are systems that contain banks of filters.



**Fig. 2.** A nonuniform filter bank system

# Introduction

- ❖ Advantages of employing filter banks
  - ✧ Different frequency bands of signals could be extracted and processed accordingly. For example, JPEG2000 employs filter banks to decompose images into different frequency bands and apply different coding schemes to code the images.
  - ✧ Enjoy advantages of time and frequency tiling because discrete wavelet transform and discrete-time wavelet transform are implemented via filter banks.

# Introduction

- ❖ Advantages of employing filter banks
  - ❧ Provide fast and low complexity implementation of signal processing because signals are decomposed into various channels and parallel processing is employed. Filter lengths are usually shorter and hence lower complexity implementation is resulted.



# Introduction

## ❖ Types of filter banks

### ∞ Uniform filter banks and nonuniform filter banks

- ❖ Uniform filter banks are filter banks that have the same samplers in all channels, while nonuniform filter banks consist of different samplers.
- ❖ Uniform filter banks can achieve perfect reconstruction easily and nonuniform filter banks may not achieve perfect reconstruction.
- ❖ Nonuniform filter banks could achieve general time frequency tiling, while uniform filter banks can only achieve uniform time frequency tiling.

# Introduction

## ❖ Types of filter banks

### ∞ Modulated filter banks

- ❖ Filters in various channels can be derived via **modulating the prototype filter**. Hence, only a single filter is required to design.
- ❖ The common types of modulated filter banks are **cosine modulated filter banks** and **discrete Fourier transform (DFT) filter banks**.
- ❖ However, frequency selectivity of the filters in the DFT filter banks could not be better than -13dB.

# Introduction

## ❖ Types of filter banks

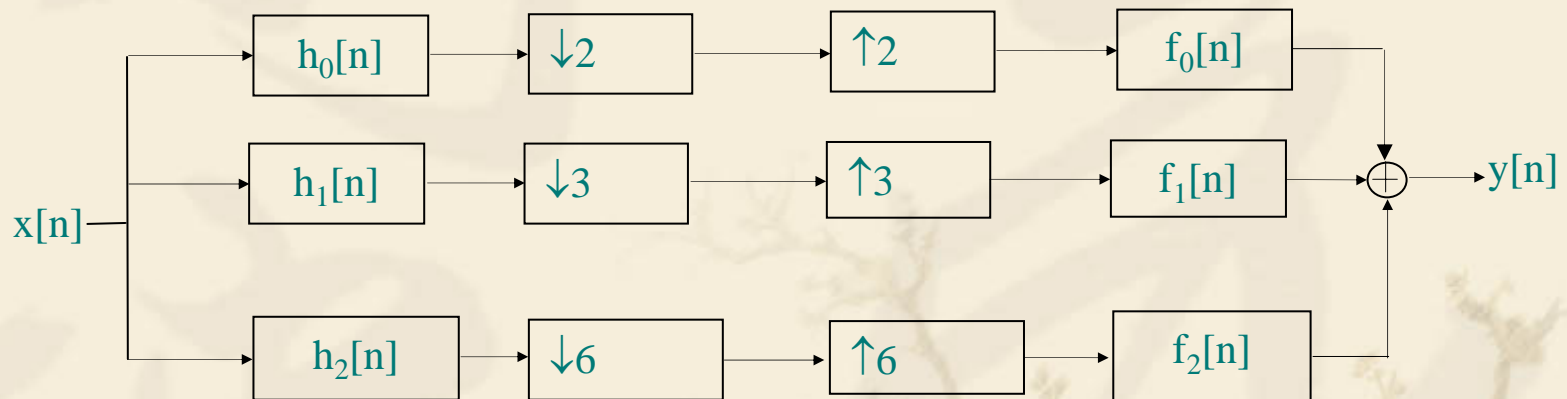
### ∞ Paraunitary filter banks

- ❖ Paraunitary filter banks are filter banks those the corresponding aliasing matrices are paraunitary.
- ❖ Tree structured paraunitary filter banks correspond to orthonormal wavelet bases.

# Introduction

## ❖ Incompatible nonuniform filter banks

⌘ Consider a nonuniform filter bank with  $M_0=2$ ,  $M_1=3$  and  $M_2=6$ .

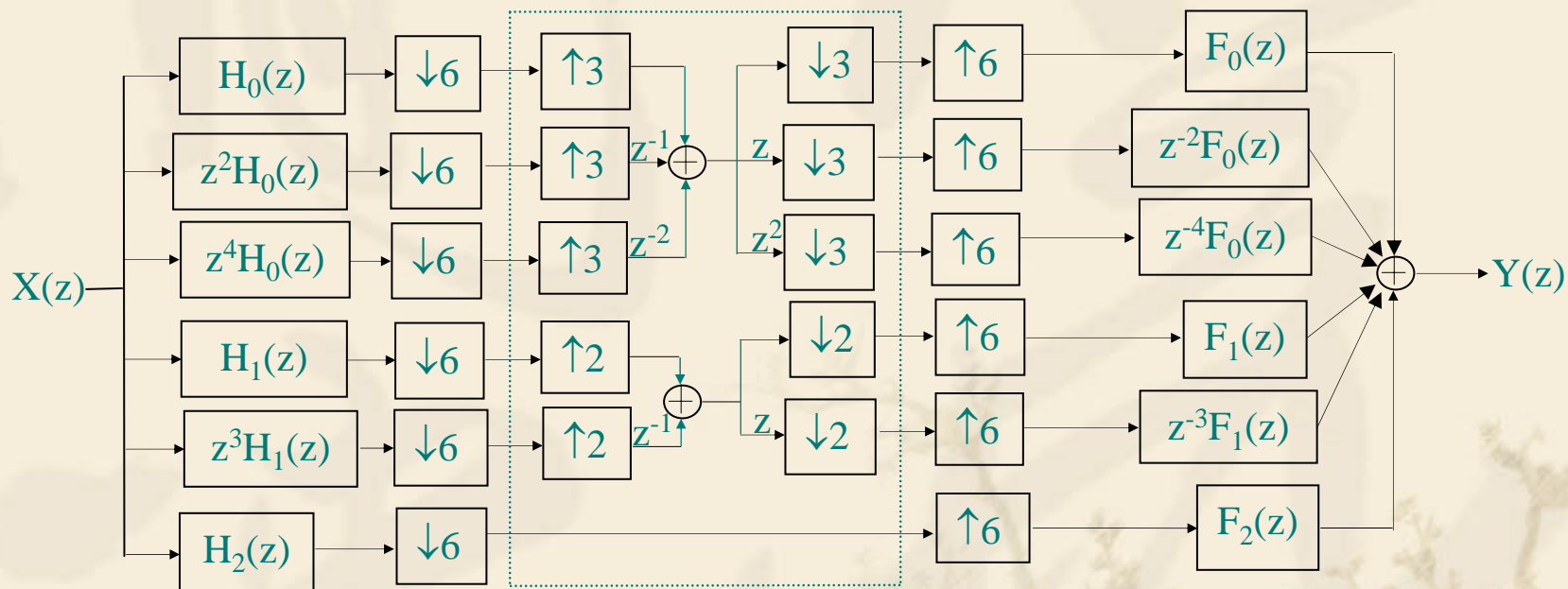


**Fig. 6.** A nonuniform filter bank system with  $M_0=2$ ,  $M_1=3$  and  $M_2=6$ .

# Introduction

## ❖ Incompatible nonuniform filter banks

☞ By converting the nonuniform filter bank to a uniform filter bank, we have:



**Fig. 7.** Converting a nonuniform filter bank system to a uniform filter bank system.

☞ The dashed rectangle is an identity system, so this nonuniform filter bank becomes a uniform filter bank.



# Introduction

## ❖ Incompatible nonuniform filter banks

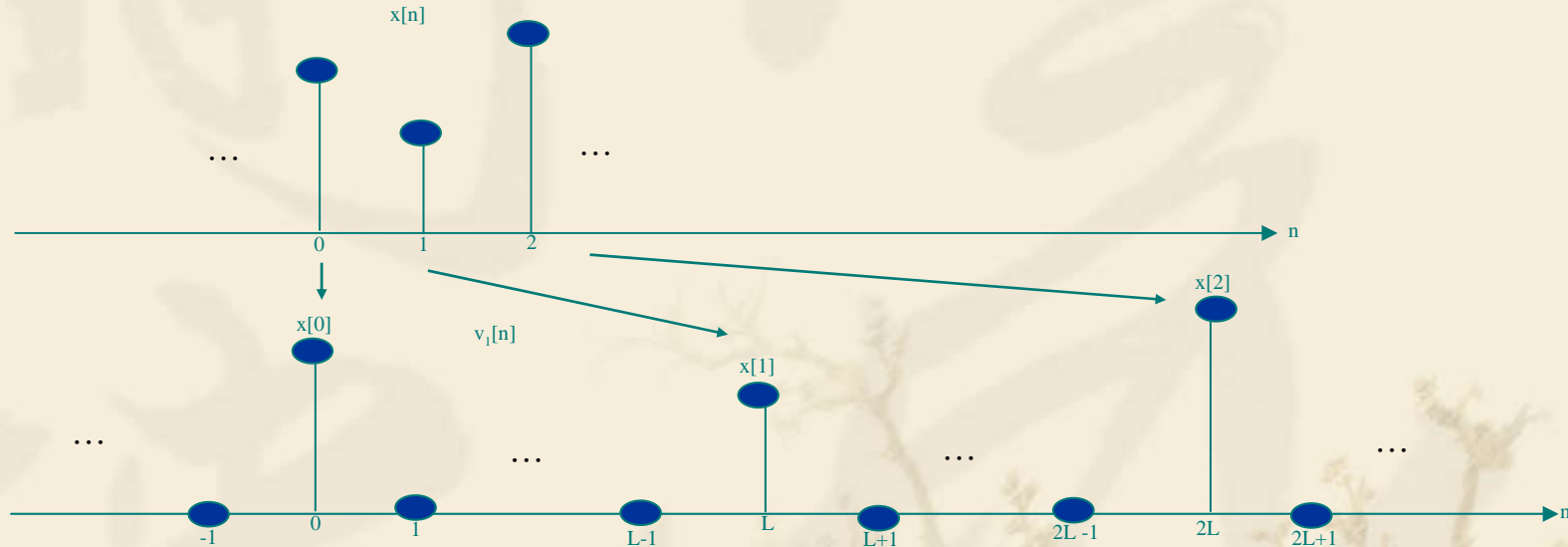
✧ It is worth noting that the second and the third synthesis filters are  $z^{-2}$  and  $z^{-4}$  related to the first filter, and the fifth filter is  $z^{-3}$  related to the fourth filter. If the corresponding uniform filter bank does not satisfy this structure, then the nonuniform filter bank cannot achieve perfect reconstruction.

# Introduction

## ❖ Incompatible nonuniform filter banks

$\propto L$ -fold expander:

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow v_1[n]$$



# Introduction

## ❖ Incompatible nonuniform filter banks

∞  $L$ -fold expander:

$$v_1[n] = \begin{cases} x\left[\frac{n}{L}\right] & \text{n is integer multiple of L} \\ 0 & \text{otherwise} \end{cases}$$

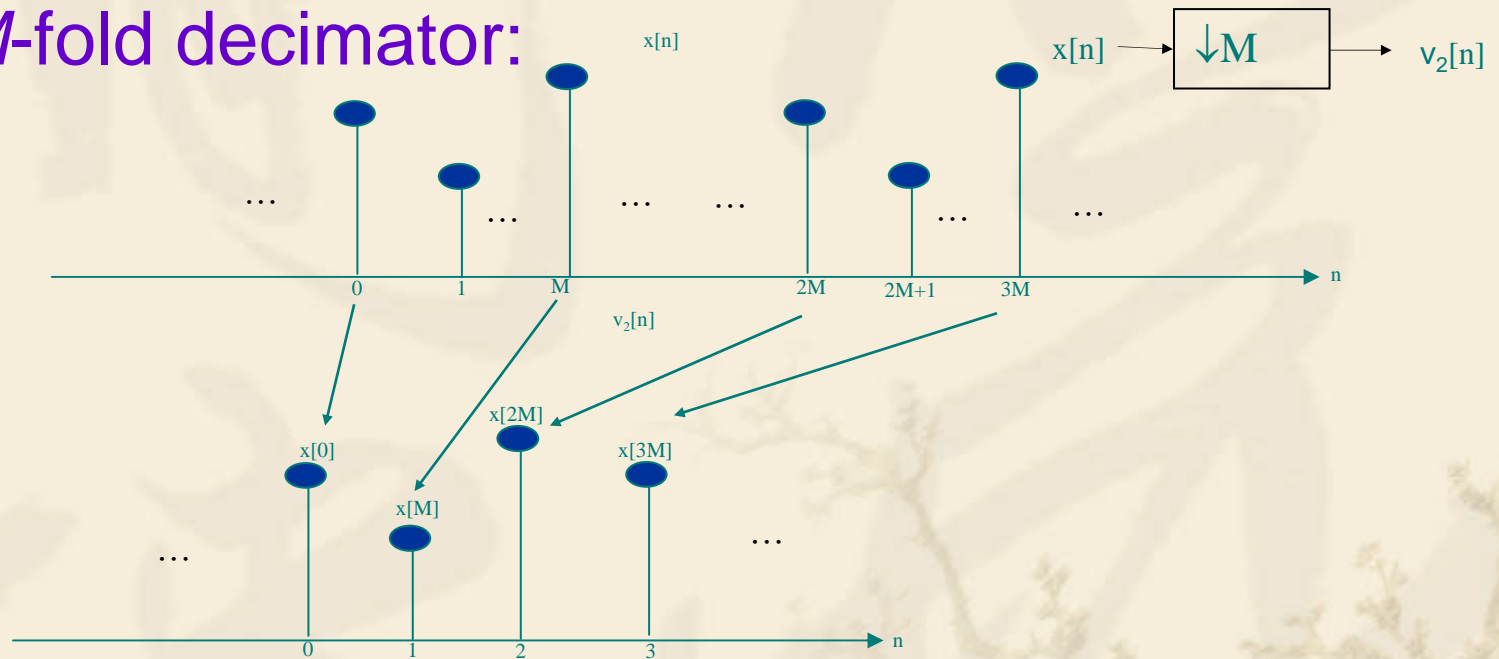
$$V_1(\omega) = X(L\omega)$$

$$V_1(z) = X(z^L)$$

# Introduction

## ❖ Incompatible nonuniform filter banks

⌘  $M$ -fold decimator:



# Introduction

## ❖ Incompatible nonuniform filter banks

∞  $M$ -fold decimator:

$$v_2[n] = x[Mn]$$

$$V_2(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{\omega - 2k\pi}{M}\right)$$

$$V_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} W^k\right)$$

where

$$W = e^{-\frac{j2\pi}{M}}$$



# Introduction

## ❖ Incompatible nonuniform filter banks

∞ Perfect reconstruction:

$$\begin{aligned}
 Y(z) &= \sum_{j=0}^{N-1} \left( \left( X(z) H_j(z) \right)_{\downarrow M_j} \right)_{\uparrow M_j} F_j(z) \\
 &= \sum_{j=0}^{N-1} \left( \frac{1}{M_j} \sum_{k_j=0}^{M_j-1} X \left( z^{\frac{1}{M_j}} W_j^{k_j} \right) H_j \left( z^{\frac{1}{M_j}} W_j^{k_j} \right) \right)_{\uparrow M_j} F_j(z) \\
 &= \sum_{j=0}^{N-1} \left( \frac{1}{M_j} \sum_{k_j=0}^{M_j-1} X(z W_j^{k_j}) H_j(z W_j^{k_j}) \right) F_j(z) \\
 &= c z^{-d} X(z) \\
 &\Rightarrow \sum_{j=0}^{N-1} \frac{F_j(z)}{M_j} \sum_{k_j=0}^{M_j-1} X(z W_j^{k_j}) H_j(z W_j^{k_j}) = c z^{-d} X(z)
 \end{aligned}$$

where  $d$  is the delay of the system,  $c$  is gain of system, and  $W_j = e^{\frac{i2\pi}{M_j}}$

# Introduction

## ❖ Incompatible nonuniform filter banks

Consider an example of  $M_0=2$ ,  $M_1=3$  and  $M_2=6$ ,  
we have:

$$\begin{bmatrix} X(z) & \dots & X(zW^5) \end{bmatrix} \begin{bmatrix} \frac{1}{2}H_0(z) & \frac{1}{3}H_1(z) & \frac{1}{6}H_2(z) \\ 0 & 0 & \frac{1}{6}H_2(zW) \\ 0 & \frac{1}{3}H_1(zW^2) & \frac{1}{6}H_2(zW^2) \\ \frac{1}{2}H_0(zW^3) & 0 & \frac{1}{6}H_2(zW^3) \\ 0 & \frac{1}{3}H_1(zW^4) & \frac{1}{6}H_2(zW^4) \\ 0 & 0 & \frac{1}{6}H_2(zW^5) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = cz^{-d} X(z)$$

where  $W = e^{-\frac{i2\pi}{6}}$

# Introduction

## ❖ Incompatible nonuniform filter banks

∞ If perfect reconstruction could be achieved for arbitrary inputs, then:

$$\begin{bmatrix} \frac{1}{2}H_0(z) & \frac{1}{3}H_1(z) & \frac{1}{6}H_2(z) \\ 0 & 0 & \frac{1}{6}H_2(zW) \\ 0 & \frac{1}{3}H_1(zW^2) & \frac{1}{6}H_2(zW^2) \\ \frac{1}{2}H_0(zW^3) & 0 & \frac{1}{6}H_2(zW^3) \\ 0 & \frac{1}{3}H_1(zW^4) & \frac{1}{6}H_2(zW^4) \\ 0 & 0 & \frac{1}{6}H_2(zW^5) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \begin{bmatrix} cz^{-d} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Introduction

## ❖ Incompatible nonuniform filter banks

- ⌘ There is only one nonzero element in the second row and the last row of the aliasing matrix.
- ⌘ This implies that  $H_2(zW)F_2(z)=0$  and  $H_2(zW^5)F_2(z)=0$ .
- ⌘ However,  $H_2(zW)F_2(z)=0$  and  $H_2(zW^5)F_2(z)=0$  implies there is no overlapping between  $H_2(\omega-\pi/3)$  and  $F_2(\omega)$ , and also between  $H_2(\omega-5\pi/3)$  and  $F_2(\omega)$ .
- ⌘ This further implies that there does NOT exist a set of nonideal filters  $\{H_0(z), H_1(z), H_2(z), F_0(z), F_1(z), F_2(z)\}$  such that  $H_2(zW)F_2(z)=0$  and  $H_2(zW^5)F_2(z)=0$ .
- ⌘ Hence, perfect reconstruction is impossible for this nonuniform filter bank.



# Introduction

## ❖ Incompatible nonuniform filter banks

- ⌘ A set of decimators is called compatible if and only if  $\forall n_i$  and  $\forall l_i \in [0, n_i - 1]$ , there exists  $n_k$  and  $l_k \in [0, n_k - 1]$  such that 
$$e^{-j2\pi \frac{l_i}{n_i}} = e^{-j2\pi \frac{l_k}{n_k}}$$
- ⌘ A nonuniform filter bank is called an incompatible if the set of decimators is not compatible.
- ⌘ An incompatible nonuniform filter bank cannot achieve perfect reconstruction.



# Introduction

- ❖ Challenges in filter banks
  - ⌘ How to achieve the perfect reconstruction for incompatible nonuniform filter banks?
  - ⌘ How to characterize an incompatible set of decimators?

# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

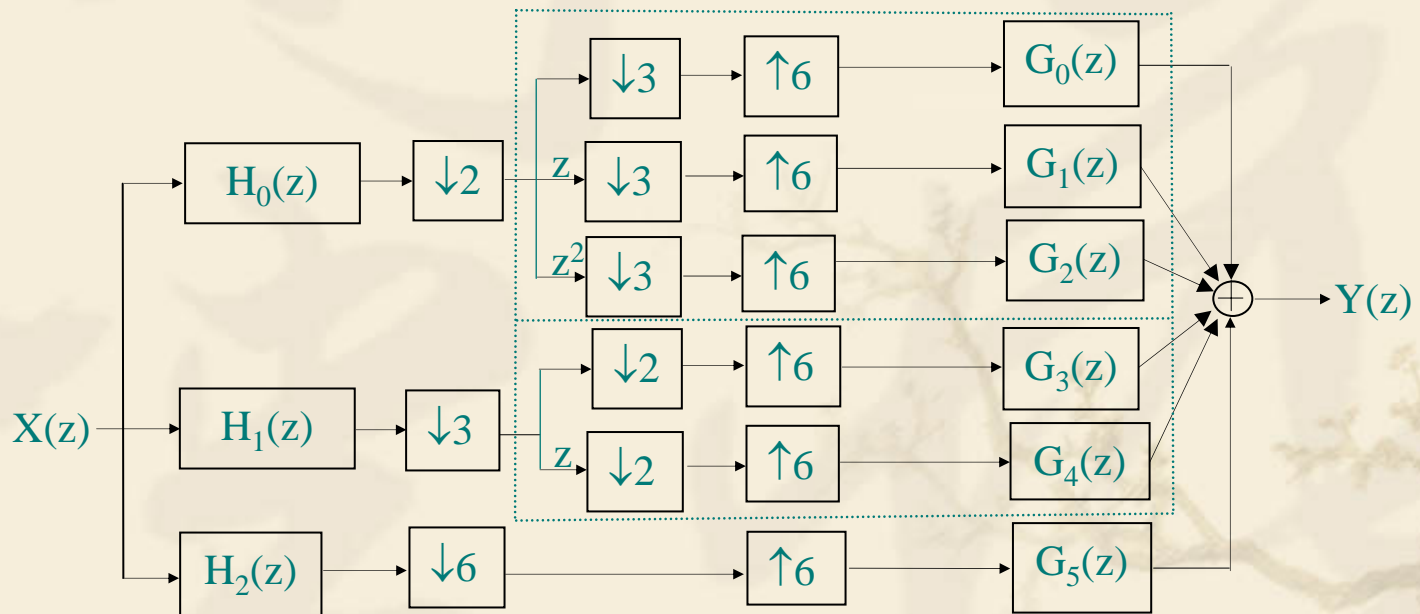
- ❖ Partial Solution
- ❖ Proposed Solution
- ❖ Characterization of incompatible sets of decimation integers

# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Partial Solution:

## Banks

☞ Convert the nonuniform filter banks to uniform filter banks.



**Fig. 10.** A modified non-uniform filter bank system which achieves perfect reconstruction

# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Partial Solution:

- ❧ Derive the set of synthesis filters in the corresponding uniform filter banks.
- ❧ The number of channels of the corresponding uniform filter bank is equal to the least common multiple of the set of decimator integers, in which it could be very large. For example, if the set of decimator integers is  $\{2, 3, 7, 43, 1807, 3263442\}$ . The 6 channel nonuniform filter bank becomes 3263442 channel uniform filter bank, which is almost impossible to realize.

# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

∞ The synthesis filter bank in the example consists of three linear dual rate systems. The output of the first linear dual rate system will shift by 6 samples if the input shifts by 3 samples. The output of the second linear dual rate system will shift by 6 samples if the input shifts by 2 samples. The output of the third linear dual rate system will shift by 6 samples if the input shifts by 1 sample.

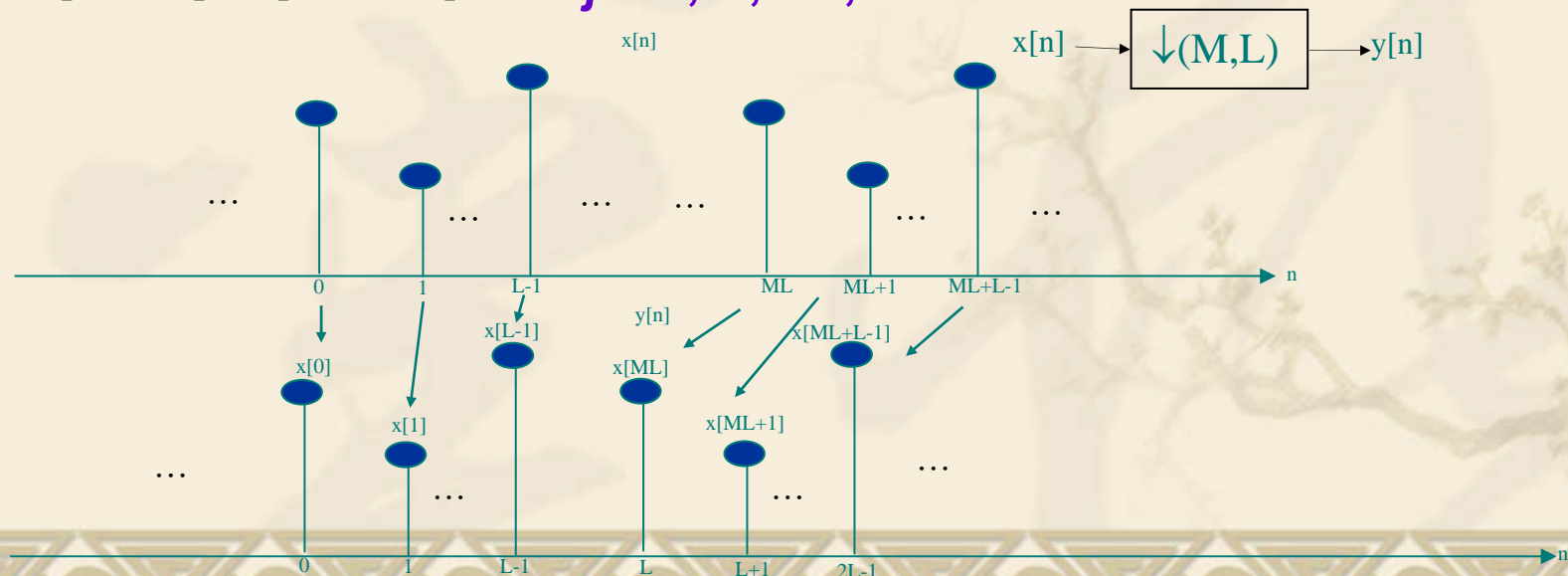


# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

⌘ Block decimators (decimation ratio  $M$  and block length  $L$ )

$$y[Lk + j] = x[kML + j] \text{ for } j=0,1,\dots,L-1 \text{ and } k \in \mathbb{Z}.$$

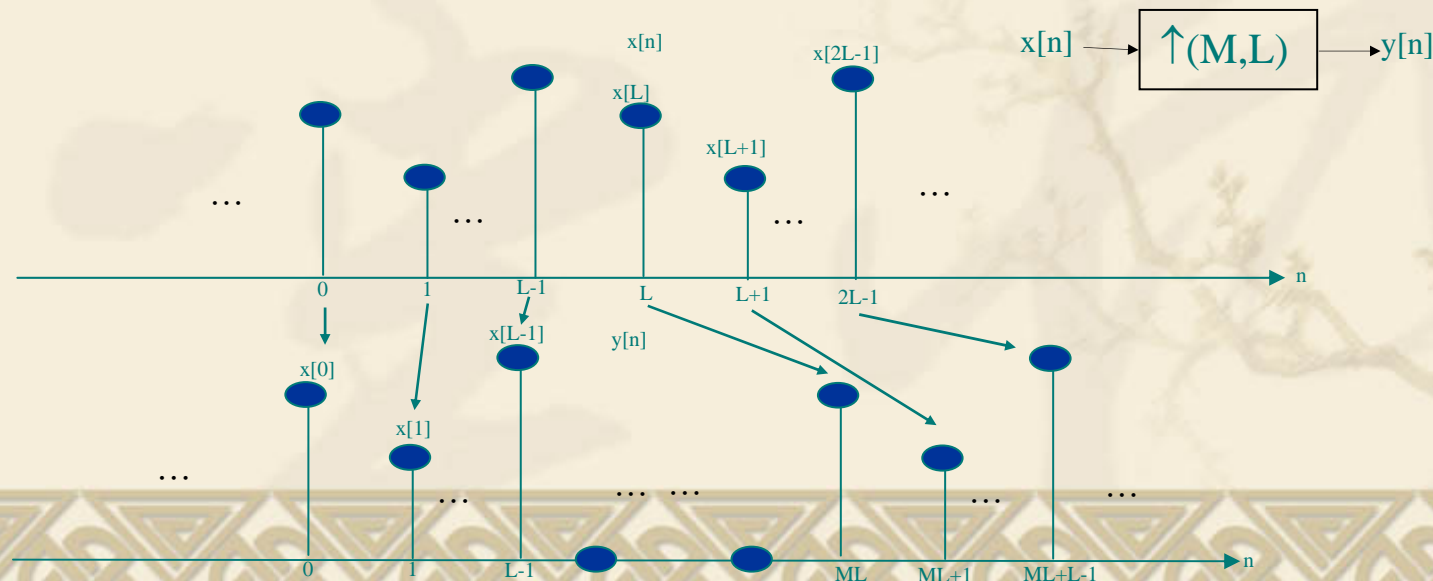


# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

↻ Block expanders (expansion ratio M and block length L)

$$y[k] = \begin{cases} x\left[\frac{k - \text{mod}(k, ML)}{M} + \text{mod}(k, ML)\right] & k - \text{mod}(k, ML) \leq k < k - \text{mod}(k, ML) + L \\ 0 & k - \text{mod}(k, ML) + L \leq k < k + ML - \text{mod}(k, ML) \end{cases}$$



# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

∞  $\forall m, n \in \mathbb{Z}^+$  (no matter  $m$  and  $n$  are co-prime or not), all linear dual rate systems with shifting input by  $n$  samples resulting to shifting an output by  $m$  samples can be represented via a series cascade of  $\uparrow m$ , followed by an LTI filter with an impulse response  $f[k]$ , and then followed by  $\downarrow(n, m)$ .

# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

∞ The input output relationship of all linear dual rate systems is  $y[km + i] = \sum_{l \rightarrow -\infty}^{+\infty} g[i, l - kn] u[l]$ ,  $\forall k, l \in \mathbb{Z}$ ,  $\forall m, n \in \mathbb{Z}^+$  and  $i = 0, 1, \dots, m-1$ .

∞ The input output relationship of the proposed representation is  $y[km + i] = \sum_{\forall l} f[kmn - ml + i] u[l]$ ,  $\forall k, l \in \mathbb{Z}$ ,  $\forall m, n \in \mathbb{Z}^+$  and  $i = 0, 1, \dots, m-1$ .

∞  $\forall k, l \in \mathbb{Z}$ ,  $\forall m, n \in \mathbb{Z}^+$  and  $i = 0, 1, \dots, m-1$ , the mapping from  $\{0, 1, \dots, m-1\} \times \mathbb{Z}$  to  $\mathbb{Z}$ , where  $[i, l - kn] \in \{0, 1, \dots, m-1\} \times \mathbb{Z}$  and  $kmn - ml + i \in \mathbb{Z}$  is bijective.

# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

- ∞ Hence,  $\forall k, l \in \mathbb{Z}$ ,  $\forall m, n \in \mathbb{Z}^+$  and  $i = 0, 1, \dots, m-1$ , there exists a unique time index  $k m n - m l + i$  corresponding to the time index  $[i, l - k n]$ .
- ∞ As a result, there exists an LTI filter with an impulse response  $f[k]$  satisfying  $f[k m n - m l + i] = g[i, l - k n]$ ,  $\forall k, l \in \mathbb{Z}$ ,  $\forall m, n \in \mathbb{Z}^+$  and  $i = 0, 1, \dots, m-1$ , that the linear dual rate systems and the proposed representation are input output equivalent.



# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

☞ Consequently, the incompatible nonuniform filter bank can achieve perfect reconstruction via the following structure:

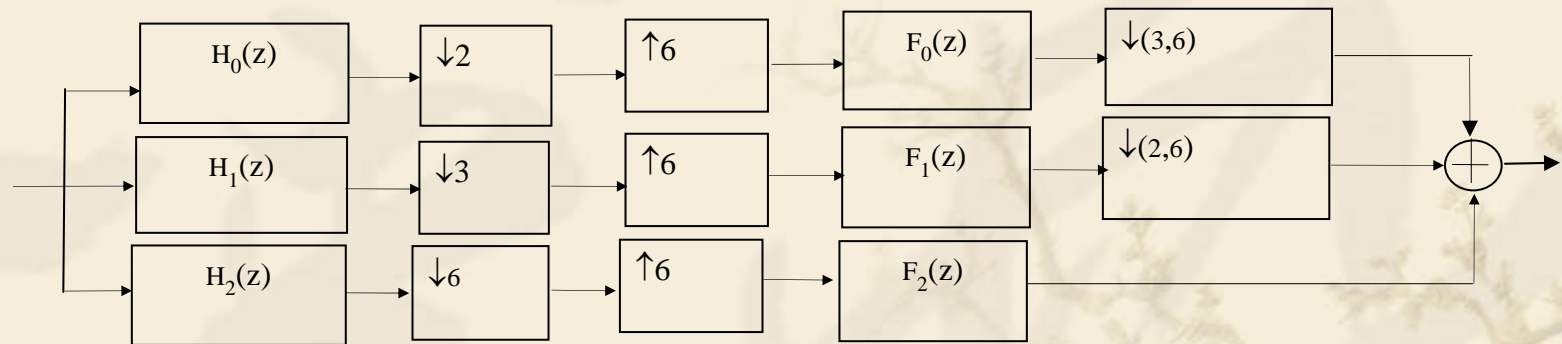


Figure 13. An incompatible nonuniform filter bank system realized via our first proposed representation.

# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

∞  $\forall m, n \in \mathbb{Z}^+$  (no matter  $m$  and  $n$  are co-prime or not), all linear dual rate systems with shifting input by  $n$  samples resulting to shifting an output by  $m$  samples can be represented via a series cascade of  $\uparrow(m, n)$ , followed by an LTI filter with an impulse response  $f[k]$ , and then followed by  $\downarrow n$ .

# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

∞ The input output relationship of all linear dual rate systems is  $y[k] = \sum_{l \rightarrow -\infty}^{+\infty} \sum_{i=0}^{n-1} g[k, nl+i] u[nl+i]$ ,  $\forall k, l \in \mathbb{Z}$ ,  $\forall m, n \in \mathbb{Z}^+$  and  $i=0,1,\dots,n-1$ .

∞ The input output relationship of the proposed representation is  $y[k] = \sum_{l \rightarrow -\infty}^{+\infty} \sum_{i=0}^{n-1} f[kn-mnl-i] u[nl+i]$ ,  $\forall k, l \in \mathbb{Z}$ ,  $\forall m, n \in \mathbb{Z}^+$  and  $i=0,1,\dots,n-1$ .

∞  $\forall l \in \mathbb{Z}$ ,  $\forall m, n \in \mathbb{Z}^+$ ,  $k \in \{0,1,\dots,m-1\}$  and  $i \in \{0,1,\dots,n-1\}$ , the mapping from  $\{0,1,\dots,m-1\} \times \mathbb{Z}$  to  $\mathbb{Z}$ , where  $[k, nl+i] \in \{0,1,\dots,m-1\} \times \mathbb{Z}$  and  $kn-mnl-i \in \mathbb{Z}$  is bijective.

# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

- ∞ Hence,  $\forall l \in \mathbb{Z}, \forall m, n \in \mathbb{Z}^+, k \in \{0, 1, \dots, m-1\}$  and  $i \in \{0, 1, \dots, n-1\}$ , there exists a unique time index  $kn - mn - l - i$  corresponding to the time index  $[k, nl + i]$ .
- ∞ As a result, there exists an LTI filter with an impulse response  $f[k]$  satisfying  $f[kn - mn - l - i] = g[k, nl + i]$ ,  $\forall k, l \in \mathbb{Z}, \forall m, n \in \mathbb{Z}^+$  and  $i = 0, 1, \dots, n-1$ , that the linear dual rate systems and the proposed representation are input output equivalent.



# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Proposed Solution:

☞ Consequently, the incompatible nonuniform filter bank can achieve perfect reconstruction via the following structure:

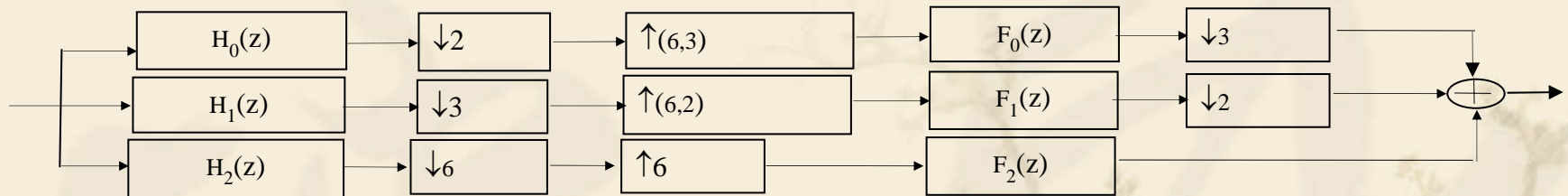


Figure 14. An incompatible nonuniform filter bank system realized via our second proposed representation.



# Perfect Reconstruction of Incompatible Nonuniform Filter Banks

## ❖ Characterization of incompatible sets of decimation integers

⌘ Denote the ordered set of maximally decimated integers as **S** such that  $\sum_{i=0}^{N-1} \frac{1}{p_i} = 1$  and  $p_i \geq p_j$  for  $i \geq j$  where  $i, j \in \{0, 1, \dots, N-1\}$ . Let the multiplicity of  $p_i$  in **S** be  $r_i$  for  $i=0, 1, \dots, N-1$ . Then **S** is incompatible if and only if  $\exists r_c = 1$  and  $p_k$  is not an integer multiple of  $p_c \ \forall k > c$ .

⌘ By characterizing the incompatible set of decimation integers, incompatible nonuniform filter banks could be avoided.

# Conclusions

- ❖ Incompatible nonuniform filter banks could achieve perfect reconstruction via block samplers.
- ❖ Incompatible sets of decimation integers are characterized so that incompatible nonuniform filter banks can be avoided.

# References

- [1] Charlotte Yuk-Fan HO, Bingo Wing-Kuen LING and Peter Kwong-Shun TAM, “Representations of Linear Dual Rate System via Single SISO LTI Filter, Conventional Sampler and Block Sampler,” to appear in IEEE Transactions on Circuits and Systems II: Transactions Briefs, 2007.
- [2] Charlotte Yuk-Fan HO, Bingo Wing-Kuen LING and Peter Kwong-Shun TAM, “Necessary and Sufficient Condition for Incompatible Set of Maximally Decimated Integers,” under revision in IEEE Transactions on Signal Processing, 2007.
- [3] Charlotte Yuk-Fan HO, Bingo Wing-Kuen LING and Zoran CVETKOVIC, “Nonuniform Filter Banks with Block Samplers Having Rational Sampling Ratios,” submitted to IEEE Transactions on Signal Processing, 2007.

# Q&A Session



Thank you!  
Let me think...

